Resonant Modes of a Stretched Wire

Object: To determine the frequency and wavelength conditions for standing wave resonance to occur in a wire.

Apparatus: Sonometer, driver and detector coils, masses, steel guitar wire, PASCO Interface and *DataStudio* Software.

Foreword

Mathematical Description of a Traveling Wave

We want to find a way to represent a transverse displacement traveling at a uniform speed v in a wire of tension T along the x-axis. First, let's agree to represent this transverse displacement by the function $Z(x, t)$. That is, the value of the function will depend on the value of x and t.

Next, let's investigate the possibility of using a function of the quantity $(x - vt)$, that is,

(1) $$Z(x, t) = f(x - vt)\ ,$$

where the choice of f corresponds to a particular moving shape. We will let $f(x - vt)$ represent the shape shown in **Figure 1**. Let's agree to pick a particular point on the function (such as the peak) and mark this point with a dot.

x	t	f(x − vt)
x_0	0	$f(x_0)$
x_1	t_1	$f(x_1 − vt_1)$
x_2	t_2	$f(x_2 − vt_2)$
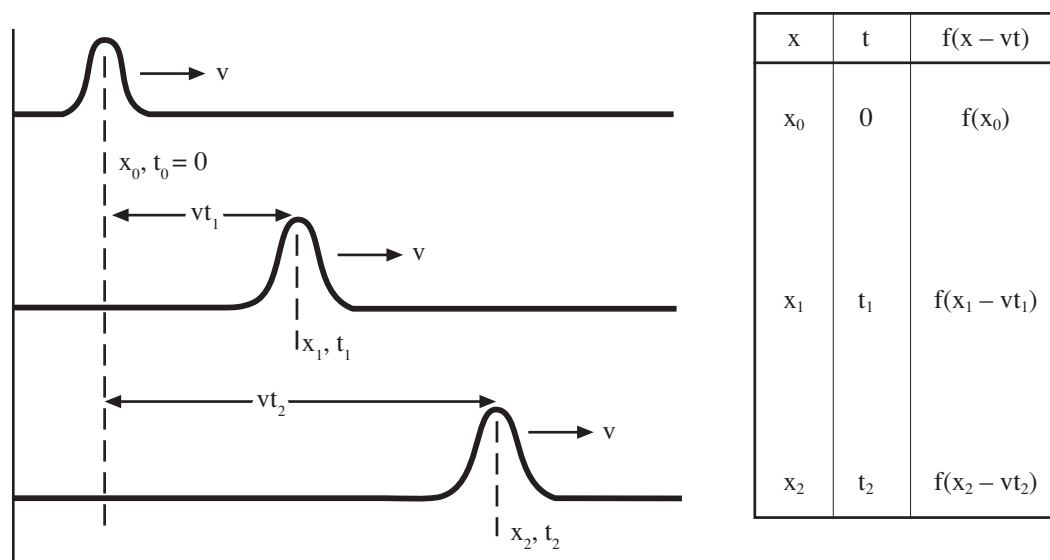
Figure 1. Transverse displacement traveling at a uniform speed v in a wire of tension T along the x-axis.

Now if the waveform is to keep the same shape, the argument which locates the dot must be the same for all values of x and t, hence:

(2)
$$x_2 - vt_2 = x_1 - vt_1$$

or

(3)
$$x_2 = x_1 + v(t_2 - t_1) \, .$$

Equation (3) describes a point moving to the right with a speed v, hence, the function $f(x - vt)$ is associated with a waveform moving to the right. In like manner, we can show that $f(x + vt)$ is associated with a waveform moving to the left with a speed v.

Since we want the waveform to be periodic, let's consider a sinusoidal form for $f(x - vt)$. For example, we might try

(4)
$$Z(x, t) = \sin(x - vt) \, .$$

We immediately see two problems with equation (4). First, the function $Z(x, t)$ represents the magnitude of the displacement of the waveform at some position x and at some time t, so it needs to have dimensions of length. Second, the argument of the sine must be dimensionless and, presently, both terms have dimensions of length.

We can take care of both of these problems by rewriting the function as

(5)
$$Z(x, t) = Z_0 \sin k(x - vt)$$

where Z_0 is the maximum value of the sine curve when $t = 0$, and k is a number which has dimension of 1/length and hence, when multiplied by $(x - vt)$, results in a dimensionless quantity for the argument of the sine. Equation (5) represents a harmonic wave traveling at a speed v in the +x direction. A harmonic wave traveling at a speed v in the –x direction is shown in equation (6).

(6)
$$Z(x, t) = Z_0 \sin k(x + vt) \, .$$

Properties of Harmonic Traveling Waves

Before moving on, let's investigate the properties of this traveling harmonic wave. Suppose we take a snapshot of $Z(x, t)$ shown in equation (5) at a time $t = 0$. The result is shown in **Figure 2**.
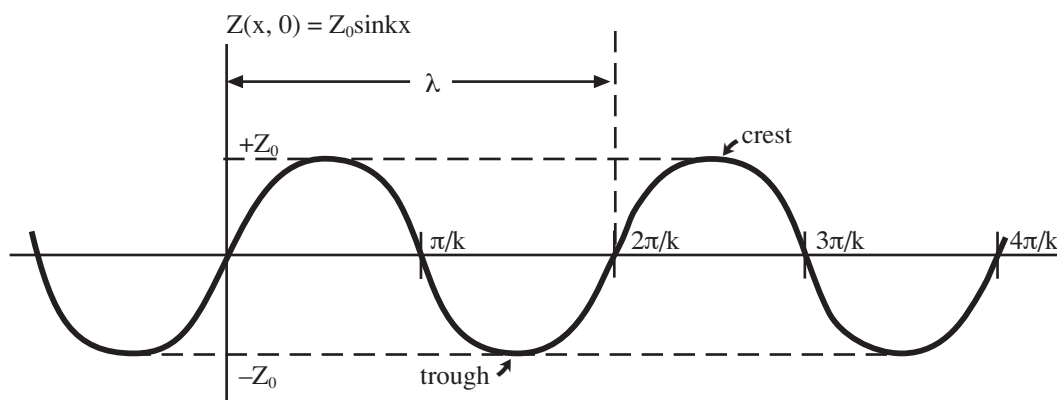
Figure 2. Harmonic traveling wave at $t = 0$.

The function $Z(x, 0) = Z_0 \sin kx$ will have a zero value whenever

(7)
$$kx = n\pi$$

or at values of x given by

(8)
$$x = n\pi/k .$$

Notice that the function repeats itself every time x increases by $2\pi/k$. Let's agree to call this repetition length the wavelength λ, hence

(9)
$$\lambda = 2\pi/k.$$

The maxima of the sine curve $(Z = +Z_0)$ are called crests and the minima $(Z = -Z_0)$ are called troughs.

From equation (3) we see that the wave form traveling at a speed v advances a distance $\Delta x = x_2 - x_1$ in a time interval $\Delta t = t_2 - t_1$. This may be written as

(10)
$$\Delta x = v\Delta t .$$

The wave form advances a distance of one wavelength $(\Delta x = \lambda)$ during a time interval of one period $(\Delta t = \tau)$. This allows us to rewrite equation (10) as

(11)
$$\lambda = v\tau .$$

Combining equations (9) and (11) we have

(12)
$$2\pi/k = v\tau$$

or

(13)
$$\tau = 2\pi/kv .$$

Recalling the relationship between frequency f and period τ

(14)
$$f = 1/\tau$$

and between frequency f and angular frequency ω

(15)
$$2\pi f = \omega$$

we may solve equation (13) for the speed of the wave and then use equations (9) and (14) to obtain:

(16)
$$v = 2\pi/k\tau = \lambda/\tau = \lambda f .$$

We may also solve equation (13) for the speed of the wave and then use equations (14) and (15) to obtain:

(17)
$$v = 2\pi/k\tau = 2\pi f/k = \omega/k .$$

Equations (16) and (17) suggest the following alternate way of writing equation (5).

$$Z(x, t) = Z_0 \sin k(x - vt)$$

$$Z(x, t) = Z_0 \sin(kx - kvt)$$

$$Z(x, t) = Z_0 \sin[(2\pi x/\lambda) - (2\pi t/\tau)]$$

(18)
$$Z(x, t) = Z_0 \sin 2\pi(x/\lambda - t/\tau)$$

Standing Waves Through Interference

In this experiment we send a harmonic wave $Z_1(x, t)$ down a wire in the $+x$ direction, and $Z_2(x, t)$ is the reflected harmonic wave in the $-x$ direction.

(19.a) $$Z_1(x, t) = Z_0 \sin 2\pi(x/\lambda - t/\tau)$$

and

(19.b) $$Z_2(x, t) = Z_0 \sin 2\pi(x/\lambda + t/\tau)$$

We are interested in the superposition of these two traveling harmonic waves.

(20) $$Z(x, t) = Z_1(x, t) + Z_2(x, t)$$

(21) $$Z(x, t) = Z_0 \sin 2\pi(x/\lambda - t/\tau) + Z_0 \sin 2\pi(x/\lambda + t/\tau)$$

Using the trigonometric identity

(22) $$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

rewrite equation (21) as

(23) $$\begin{aligned} Z(x, t) = Z_0[&\sin(2\pi x/\lambda)\cos(2\pi t/\tau) - \cos(2\pi x/\lambda)\sin(2\pi t/\tau) \\ &+ \sin(2\pi x/\lambda)\cos(2\pi t/\tau) + \cos(2\pi x/\lambda)\sin(2\pi t/\tau)] \end{aligned}$$

which reduces to

(24) $$Z(x, t) = 2Z_0 \sin(2\pi x/\lambda)\cos(2\pi t/\tau) .$$

This waveform has some interesting characteristics. At some time t_0 we get

(25) $$Z(x, t_0) = [2Z_0 \cos(2\pi t_0/\tau)]\sin(2\pi x/\lambda) .$$

So, the shape of the wire (that is, the x dependence) will be a sine curve $[\sin(2\pi x/\lambda)]$, and the amplitude at any given point x_0 is periodic and determined by $[2Z_0\cos(2\pi t_0/\tau)]$.

Lets look at points along the wire where

$$(2\pi/\lambda)x = n\pi/2 \qquad\qquad (n = 0, 1, 2, 3, 4, 5, ...)$$

or

$$x = n(\lambda/4) \qquad\qquad (n = 0, 1, 2, 3, 4, 5, ...)$$

Table 1 below summarizes the results for different values of n.

n	$x = n\lambda/4$	$\sin(2\pi x/\lambda)$ $= \sin(n\pi/4)$	$Z(x, t_0) =$ $[2Z_0\cos(2\pi t_0/\tau)]\sin(2\pi x/\lambda)$
0	0	0	0
1	$\lambda/4$	$+1$	$+2Z_0\cos(2\pi t_0/\tau)$
2	$\lambda/2$	0	0
3	$3\lambda/4$	-1	$-2Z_0\cos(2\pi t_0/\tau)$
4	$\lambda/$	0	0
5	$5\lambda/4$	$+1$	$+2Z_0\cos(2\pi t_0/\tau)$
6	$3\lambda/2$	0	0
7	$7\lambda/4$	-1	$-2Z_0\cos(2\pi t_0/\tau)$
8	2λ	0	0

Table 1. A summary of the values of $Z(x, t_0)$ for various values of x.

We see from **Table 1** that the value of $Z(x, t_0)$ will be zero at $x = 0$ and every half wavelength after that. These points are called nodes and are points where destructive interference is occurring. We also note from **Table 1** that the value of $Z(x, t_0)$ will be a maximum at one-fourth wavelength and every half wavelength after that. These points are called antinodes and are points where constructive interference is occurring.

The results of **Table 1** are shown in **Figure 3**.
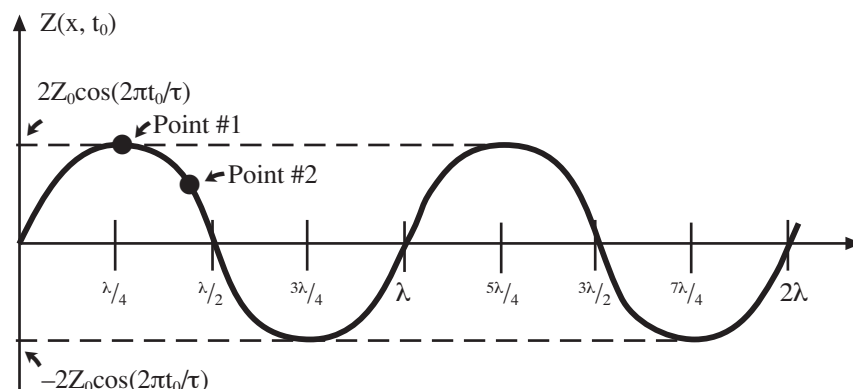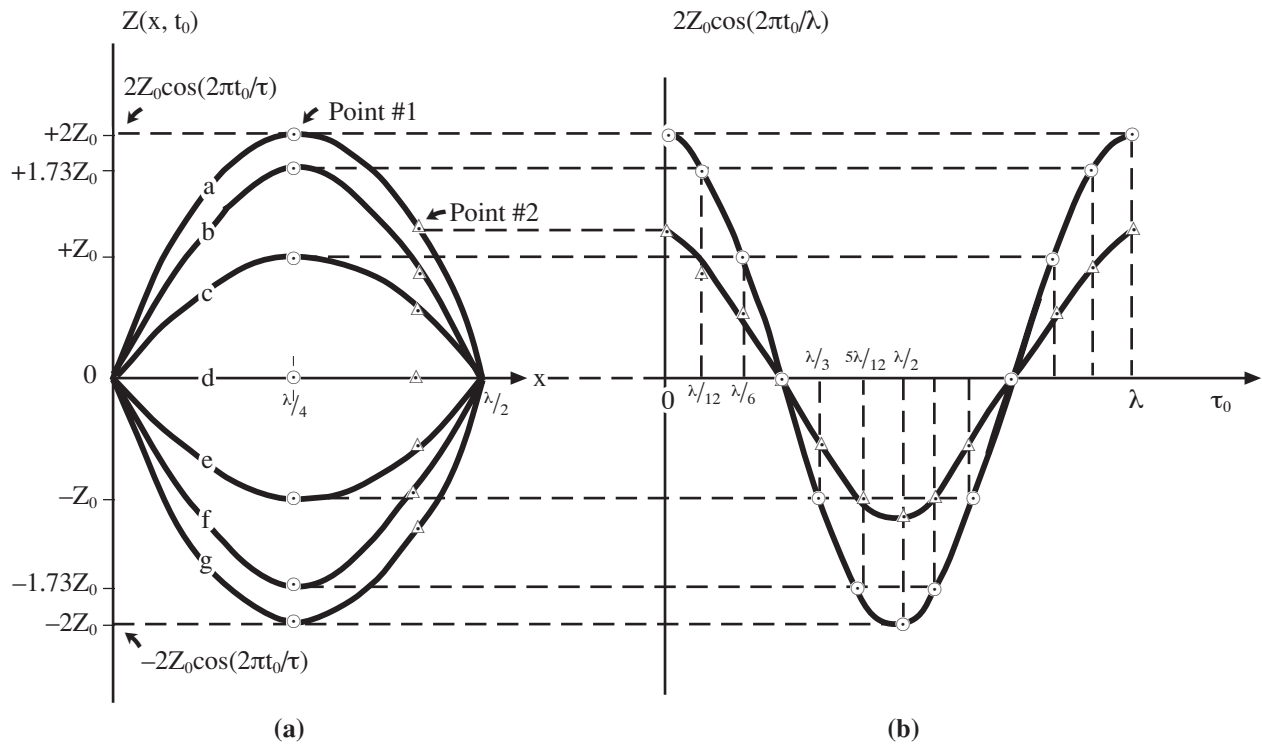

Figure 3

To further see how the wire shown in **Figure 3** is behaving, let's look at the region $x = 0$ to $x = \lambda/2$ for a time interval long enough for this segment of the wire to go through a complete cycle of its motion. **Figure 4** shows what this section of the wire looks like at a number of different times. The motion of two different points (Point 1 and Point 2) is then viewed over the time for one complete cycle.

Table 2 below shows the value of amplitude maximum at several different times and tells which sketch of **Figure 4** shows the wave at that time.

t_0	$2Z_0\cos(2\pi t_0/\lambda)$	Sketch in Fig. 4
$0, \tau$	$2Z_0(1) = 2Z_0$	a
$\frac{1}{12}\tau, \frac{11}{12}\tau$	$2Z_0(0.866) = 1.73Z_0$	b
$\frac{1}{6}\tau, \frac{5}{6}\tau$	$2Z_0(0.500) = Z_0$	c
$\frac{1}{4}\tau, \frac{3}{4}\tau$	$2Z_0(0) = 0$	d
$\frac{1}{3}\tau, \frac{2}{3}\tau$	$2Z_0(-0.500) = -Z_0$	e
$\frac{5}{12}\tau, \frac{7}{12}\tau$	$2Z_0(0.866) = -1.73Z_0$	f
$\frac{1}{2}\tau$	$2Z_0(1) = -2Z_0$	g

Table 2. Maximum value of the amplitude of the wave form shown in **Figure 4** at various times.

(a) $x = 0$ to $x = \lambda/2$ segment of the wire at seven different positions (a, b, c, d, e, f, and g)

(b) Trace of positions #1 and #2 of the wire for a time interval of one period.

Figure 4

Note that the motion of a given point is periodic with its maximum amplitude determined by its x position. Now let's look again at the function $Z(x, t)$ describing the amplitude of the waveform.

At a fixed position x_0 on the wire, the wire is undergoing simple harmonic motion with an amplitude of

$$\text{(26)} \qquad Z(x_0, t) = (2Z_0\sin 2\pi x_0/\lambda)\cos 2\pi t/\tau$$

So the amplitude at some point x_0 varies in time as a cosine function with maximum amplitude of $2Z_0\sin(2\pi x_0/\lambda)$ and a zero value when:

$$\text{(27)} \qquad\qquad 2\pi t/\tau = n\pi/2 \qquad\qquad n = 1, 3, 5, ...$$

or

$$\text{(28)} \qquad\qquad t = n\tau/4 \qquad\qquad n = 1, 3, 5, ...$$

All of the above is shown in **Figure 4**.

This waveform is called a standing wave. There is no propagation of the waveform along the wire. Each point of the wire oscillates up and down with its amplitude determined by $2Z_0\cos(2\pi t/\tau)$. The points of maximum amplitude are called antinodes and the points of zero amplitude are called nodes.

Resonance

The preceding analysis assumes the standing wave

(29) $$Z(x, t) = (2Z_0\sin 2\pi x/\lambda)\cos 2\pi t/\tau$$

is formed by the superposition of an original wave and one reflected wave. If the string is fixed at both ends, each wave will be reflected every time it reaches either end of the string. In general, the repeatedly reflected waves will not all be in phase, hence the amplitude of the wave pattern will be small. However, at certain frequencies of oscillation, all the reflected waves will be in phase, resulting in a very high amplitude standing wave. These frequencies are called **resonant frequencies**.

In this experiment, the relationship between the length of the string and the frequencies at which standing wave resonance occurs is investigated. Since the conditions for resonance are more easily pictured in terms of the wavelength, we will find it convenient to picture the resonant modes in terms of the wavelength of the wave pattern, rather than in terms of the frequency. In general, standing wave resonance will occur when the wavelength satisfies the condition:

(30) $$\lambda = 2L/n \qquad (n = 1, 2, 3, ...)$$

Another way of stating this same thing is to say that standing wave resonance occurs when an integral number of half wavelengths of the wave pattern under investigation fit into the length of the wire with a node at each end.

Procedure

In this activity, a Driver Coil connected to the Power Amplifier is used to vibrate a thin steel wire that is stretched over two "bridges" on a Sonometer. The Signal Generator in **DataStudio** controls the frequency at which the wire is vibrated. A Detector Coil monitors the amplitude and frequency of the vibrating wire.

The **DataStudio** program displays the output signal from the signal generator that controls the Driver Coil, and the input signal from the Detector Coil. You will determine the resonant frequencies of the stretched wire by watching the amplitude of the input signal.

I. Computer and Interface Setup

1.   Make sure the Pasco interface is connected to the computer, turn on the interface, and turn on the computer.

2.   Connect the Power Amplifier to Analog Channel A of the interface.

3.   Connect the Voltage Sensor DIN plug to Analog Channel B of the interface.
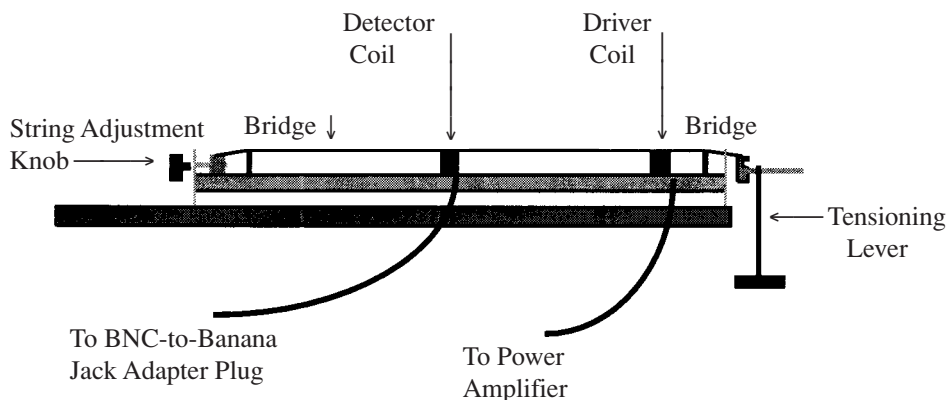
II. Software Setup

1.   Open the software package titled **DataStudio**.

2.   Select the *"**Power Amplifier**"* icon from the list of sensors and drag it to Analog Channel A.

3.   When you "plug in" the Power Amplifier icon, a *"**Signal Generator**"* window will appear. Set the Signal Generator to output a *"**Sine Wave Function**"* at **4.0 Volts**, and let's agree to start with a frequency of **50Hz**.
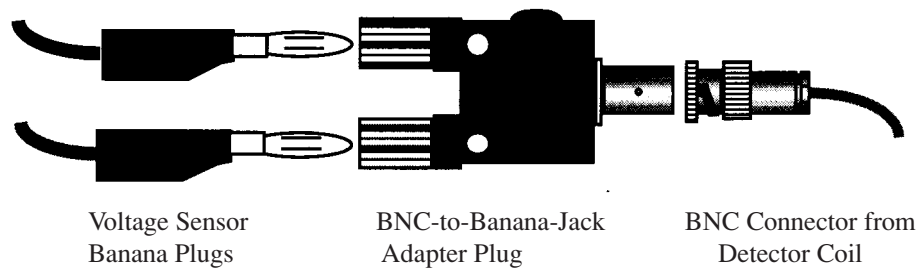
4. For now, we don't want the Signal Generator to come on automatically, so de-select this option by clicking on the "***AUTO***" button on the Signal Generator.

5. Select the "***Voltage Sensor***" icon from the list of sensors and drag it to Analog Channel B.

6. Drag the "***Scope***" icon to the Signal Generator window and release.

7. Drag the "***Voltage, Ch.B***" icon from the DATA window to the middle of the oscilloscope window and release.

   By playing with the scope, you can determine how to adjust the scale, sensitivity, and location for each trace in both the X and Y direction.

8. Adjust the scale for the "***Voltage, Ch.B***" trace to "***0.1V/div***".

9. Drag the "***FFT***" *(Fast Fourier Transform)* icon to the "***Voltage, Ch.B***" icon and release.

10. At this point it will be convenient to close the interface display and hide the Summary windows on the left by clicking on the "***Summary***" button on the main Toolbar. Position things on the screen so that you can see the function generator, the O-scope, and the FFT.

III. Sonometer Setup

1. Set the Sonometer at the edge of a table so the tensioning lever extends beyond the table.

2. Start with the bridges 60 cm apart. Make sure the wire is attached to the peg on the cylinder with the string adjustment knob, and to the rounded slot in the vertical section of the tensioning lever.

3. Hang a 1 kg mass from the **second** notch in the tensioning lever (counting starts closest to the Sonometer). Use the string adjustment knob to tighten or loosen the wire until the tensioning lever is horizontal.

4. Position the Driver Coil approximately 5 cm from the bridge nearest the Tensioning Lever.

5. Connect the Driver Coil banana plugs into the Power Amplifier output jacks.

| Voltage Sensor | BNC-to-Banana-Jack | BNC Connector from |
| Banana Plugs | Adapter Plug | Detector Coil |

6.   Position the Detector Coil near the center of the wire between the two bridges. Attach the BNC connector on the Detector Coil cable to the BNC-to-banana-jack Adapter Plug. Connect the Voltage Sensor banana plugs into the jacks on the Adapter Plug.

7.   Calculate the tension in the wire by multiplying the number of the notch on the tensioning lever by the weight of the hanging mass (mass (kg) x 9.8 N/kg).

8.   Record the length, tension, and linear density of the wire in the Data section.

IV. Determining the Frequency of the First Resonant Mode (the Fundamental Frequency)

We will use the Frequency Spectrum [Fast Fourier Transform (FFT)] display to measure the approximate value of the fundamental frequency of the wire on the Sonometer. **NOTE:** *The fundamental frequency is the frequency at which the wire vibrates when it is plucked.*

1.   Click the "*START*" button to begin monitoring data.

2.   Pluck the wire near the center of its length. As the wire vibrates, the Frequency Spectrum (FFT) display will show the fundamental frequency recorded by the detector coil.   MAKE NOTE OF THE FUNDAMENTAL FREQUENCY.

3.   Click the "*Stop*" button to stop monitoring data.

> *Now that we have a rough estimate of the fundamental frequency, we can use that information as a starting point for investigating the exact fundamental frequency. We can do this because we know that when we drive the string at a resonant frequency, such as the fundamental frequency, the string will oscillate visibly, AND we will be able to see maximum amplitude for the string's vibration on the oscilloscope.*

4.   Click the Signal Generator window to make it active and click the *"AUTO"* button so that it'll automatically turn on when we start gathering data again.

5.   Set the frequency in the Signal Generator window at a value that is approximately **one-half of the fundamental frequency of the wire**. *(For example, if the approximate fundamental frequency is 110 Hz, set the frequency in the Signal Generator window to 55 Hz.)*

NOTE: The reason that the driving frequency in the Signal Generator should be approximately one-half of the fundamental frequency is because the driver coil (an electromagnet) pulls on the metal wire TWICE per cycle. Therefore, if you set the driving frequency at 60 Hz, the wire will vibrate at 120 Hz. See Experiment 2W for more details behind the Physics of this factor of two.

6.   Click *"Start"* to begin monitoring data again. Observe the middle area of the wire, and the traces on the Scope display.

7.   Slightly adjust the signal frequency up and down to begin searching for the exact fundamental frequency. Watch the wire and the traces on the Scope.

8.   When you are satisfied that the wire is in its fundamental resonant mode, record the Signal Generator frequency at the appropriate location on the Data Sheet.

V. Determining the Frequency of the Higher Resonant Modes

1.   Slide the detector coil away from the middle so the detector coil is at a position about one-fourth of the distance between the bridges.

2.   Set the Signal Generator frequency to <u>twice</u> its original frequency. Observe the vibrating wire.

3.   Adjust the frequency up and down until you are satisfied that the wire is in its second resonant mode.

4.   Record the Signal Generator frequency.

   • *Remember, the wire is vibrating at a frequency that is twice the frequency of the Signal Generator.*

5.   Slightly adjust the position of the detector coil back and forth. Observe the change in amplitude of the trace on the Scope display. When the detector is beneath an antinode the voltage from the detector is at a maximum amplitude and when the detector is beneath a node the voltage from the detector is at its minimum amplitude. When you are satisfied that the detector coil is beneath a node on the wire, record the position of the detector coil. Record the position of all the nodes (including the nodes at the bridges) and then determine and record the positions of the antinodes.

6.   Repeat the procedure of modifying the frequency to find the higher resonant modes. Record the Signal Generator frequencies and the distances between adjacent antinodes for the next three resonant modes. Remember, you can get the distances between adjacent antinodes by sliding the detector coil from one antinode to the next antinode.

Conclusions

1. Draw a figure of a wire showing each resonant waveform which you investigated and then make concluding remarks regarding the following:

    (a) The relationship between the number of antinode segments and the number of the resonant mode?

    (b) The relationship between the resonant frequency and wavelength.

2. Note from your data and calculation sheet that $f_{wire}\lambda$ is a constant. Make concluding remarks regarding the physical significance of this constant. As a check on your thoughts, determine this constant by some other method and then compare your answers.

3. Note from your data and calculation sheet that $n\lambda/2$ is a constant. Make concluding remarks regarding the physical significance of this constant. As a check on your thoughts, determine this constant by some other method and then compare your answers.

4. Make concluding remarks regarding the mathematical relationship between the lowest resonant frequency and the higher frequencies at which resonance occurs.

Questions

1. Suppose you keep the linear mass density and the tension of the wire the same but you decrease the length of the wire by a factor of two. Compare the value of the frequencies for the resonant modes for this new situation to the value of the frequencies for the resonant modes for the original situation.

2. Suppose you keep the linear mass density and the length of the wire the same but you increase the tension in the wire by a factor of four. Compare the value of the frequencies for the resonant modes for this new situation to the value of the frequencies for the resonant modes for the original situation.

Exploration

1. Keep the tension in the wire and the length of the wire the same but increase the linear mass density of the wire. Using your data, theoretically predict the fundamental frequency for this new situation and then experimentally check your results.

2. Change the string tension by hanging the weight from a different notch. Experiment as needed to answer the following questions. Do the frequencies at which resonance occurs depend on the tension of the wire? Do the shapes of the resonance patterns (locations of nodes and antinodes) depend on the tension of the wire?

DATA AND CALCULATION SHEET

Distance between bridges   = _____ m = Length of the wire under investigation

Linear mass density of wire   = _____ kg/m

Tension in the wire   = _____ N

Approximate fundamental frequency from "plucking" the wire = _____ Hz

Resonate Mode	First	Second	Third	Fourth	Fifth
Function Generator Frequency (Hz)					
Wire Frequency (H z)					
Distance between nodes (m)					
Number of Antinodes					
Number of Half Wavelengths					
Half Wavelength $\lambda/2$ (m)					
Wavelength λ (m)					
$f_{wire}\lambda$ (m / s)					
$n\lambda/2$ (m)					